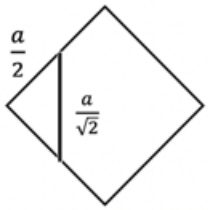
SOLUTION T1 [10 pts]. Three problems

T1-A [3.0 pts]. Hydrostatic gate

T1-A1 [3.0 pts].
Since the dam is perfectly sealed, the area of the opening is $a \cdot (a/\sqrt{2})$:



The horizontal force exerted by the water on either side is equal to the pressure at the centroid of the cube sections on the corresponding side times the effective area, as calculated previously:

$$\Delta F = \frac{a^2}{\sqrt{2}}\rho_0 g \Delta h$$

With an effective application point at the centroid of the left region of the block. The other relevant forces are the block's weight $\rho \times g \times vol$, the vertical component of the force exerted by the fluid (buoyant force) $\rho_0 \times g \times vol$, and the force that the hinge applies to the block (which exerts no torque).
At the critical Δh, the bottom edge of the opening applies zero force to the block. For the net torque to be 0,

$$\frac{a^2}{\sqrt{2}}\rho_0 g \Delta h \left(\frac{a}{2\sqrt{2}}\right) = (\rho - \rho_0)a^3 g \left(\frac{a}{2\sqrt{2}}\right)$$

Note: Arms are $a/(2\sqrt{2})$ from O either in the horizontal or vertical direction for pressure difference and effective weight respectively.

$$a = \frac{\rho_0}{\sqrt{2}(\rho - \rho_0)}\Delta h$$

For $\rho = 3\rho_0$,

$$\boxed{a = \frac{\Delta h}{2\sqrt{2}} = 0.50m}$$

T1-B [2.5 pts]. Electron-positron pair

From the definition of angular momentum,

$$mv_0(50a_0) = \mu\hbar$$

$$v_0 = \frac{\mu\hbar}{50ma_0}$$

The total energy of the system is

$$2\left(\frac{1}{2}mv_0^2\right) - \frac{ke^2}{100a_0} = \frac{ke^2}{a_0}\left(\frac{\mu^2}{50^2} - \frac{1}{100}\right)$$

T1-B1 [1.0 pts].

For $\mu = 4$,

$$E = -\frac{9ke^2}{2500a_0}$$

SOLUTION 1:
At the critical separation distances, $\vec{v} \perp \vec{r}$ and $L = mvr$. The energy at r_{max} and r_{min} can be written as

$$E = 2\frac{L^2}{2mr^2} - \frac{ke^2}{r}$$

Using $L = 2\mu\hbar = 8\hbar$ and the conservation of energy,

$$-\frac{9ke^2}{2500a_0} = \frac{64\hbar^2}{mr^2} - \frac{ke^2}{r}$$

$$\frac{ke^2}{a_0}\left(\frac{9}{2500}r^2 - a_0 r + 64a_0^2\right) = 0$$

$$r = \frac{a_0 \pm \sqrt{a_0^2 - \frac{576}{625}a_0^2}}{9/1250}$$

$$r = \frac{1250}{9}a_0\left(1 \pm \frac{7}{25}\right)$$

The maximum separation corresponds to the solution with the plus sign,

$$\boxed{r_{max} = \frac{1600}{9}a_0}$$

SOLUTION 2:
Since the energy is negative and the electrostatic force is central and proportional to r^{-2}, the trajectory of the particles will be elliptical with respect to the CM. The energy can be written in terms of the semi-major axis x as

$$E = -\frac{ke^2}{2x}$$

$$x = \frac{1250}{9}a_0$$

Since $\vec{v}_0$ is perpendicular to the vector that connects the two particles, the initial position must correspond to either r_{max} or r_{min}. The other critical distance, r_2 is given by the definition of x:

$$2x = r_0 + r_2$$

$$\boxed{r_2 = \left(\frac{2500}{9} - 100\right)a_0 = \frac{1600}{9}a_0}$$

Which is greater than r_0 and corresponds to r_{max}

SOLUTION 3:
The general equation for the trajectory in polar coordinates is

$$r(\theta) = \frac{\ell}{1 + \varepsilon\cos(\theta - \theta_0)}$$

with $\ell = \frac{2L^2}{mke^2}$ the semi-latus rectum and ε the eccentricity, as given by the hint in question T1-B2. The maximum separation distance is reached when $\cos(\theta - \theta_0) = -1$:

$$r_{max} = \frac{\ell}{1 - \varepsilon}$$

Calculating the values of ℓ and ε,

$$\ell = \frac{2(8\hbar)^2}{mke^2} = 128a_0$$

$$\varepsilon = \sqrt{1 - \frac{36ke^2(8\hbar)^2}{2500a_0m(ke^2)^2}}$$

$$\varepsilon = \sqrt{1 - \frac{2304}{2500}} = \frac{7}{25}$$

Replacing these values in the expression for r_{max},

$$\boxed{r_{max} = \frac{128}{1 - (7/25)}a_0 = \frac{1600}{9}a_0}$$

T1-B2 [2.5 pts].
For $\mu = 15/2$,

$$E = +\frac{ke^2}{80a_0}$$

The distance between particles follows

$$r(\theta) = \frac{\ell}{1 + \varepsilon\cos(\theta - \theta_0)}$$

Since the initial position corresponds to the closest approach, it is convenient to measure the angle θ with respect to the y axis so that $\cos\theta_0 = 1$, $\theta_0 = 0$. For $r \to \infty$,

$$1 + \varepsilon\cos\theta' \to 0$$

$$\theta' = \arccos(-1/\varepsilon)$$

Calculating the eccentricity,

$$\varepsilon = \sqrt{1 + \frac{4ke^2(15\hbar^2)}{80a_0m(ke^2)^2}}$$

$$\varepsilon = \sqrt{1 + \frac{45}{4}} = \frac{7}{2}$$

Replacing this value in the expression for θ,

$$\theta' = \arccos(-2/7) = 106.60^\circ$$

Since this is the angle with respect to the y-axis but the initial e^+ velocity is directed towards the positive x-axis, heading downward

$$\boxed{\theta = 90^\circ - \theta' = -16.60^\circ}$$

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T1-C [3.5 pts]. Photodissociation of ozone

T1-C1 [2.5 pts].

SOLUTION 1: pure algebra

The momentum conservation is

$$\hbar\frac{\omega}{c}\hat{k} = \vec{p} + \vec{q},$$

where $\vec{p}$ and $\vec{q}$ are the momentum of the O_2 and the O, respectively. Solving for q^2,

$$q^2 = \left(\frac{\hbar\omega}{c} - p\cos(\theta)\right)^2 + p^2\sin^2(\theta).$$

The energy conservation reads

$$\hbar\omega = \Delta U + \frac{p^2}{4m} + \frac{q^2}{2m}$$

where $\Delta U = U_f - U_i$.

For a given θ, p can be written in terms of ω. Replacing q^2 on energy balance and simplifying, we get

$$p^2 - \frac{4\hbar\omega\cos(\theta)}{3c}p + \frac{2\hbar^2\omega^2}{3c^2} + \frac{4}{3}m(\Delta U - \hbar\omega) = 0.$$

This gives a quadratic equation on p. For p to be real, the discriminant must be positive.

$$\frac{16\hbar^2\omega^2\cos^2(\theta)}{9c^2} - 4\left(\frac{2\hbar^2\omega^2}{3c^2} + \frac{4}{3}m(\Delta U - \hbar\omega)\right) \geq 0$$

The previous bound determines $\omega_{\min}$ as the minimal root of the quadratic equation on ω

$$(\cos(2\theta) - 2)\hbar^2\omega^2 + 6mc^2\hbar\omega - 6\Delta U mc^2 = 0$$

$$\omega_\pm = \frac{-6mc^2\hbar \pm \sqrt{D}}{2\left(\cos(2\theta) - 2\right)\hbar^2}$$

$$D = 36\hbar^2m^2c^4 + 24\Delta U mc^2\hbar^2\left(\cos(2\theta) - 2\right)$$

The smallest root ω_+. Thus, simplifying, one gets

$$\boxed{\omega_{\min} = \frac{3mc^2 - \sqrt{3mc^2\left(3mc^2 + 2\Delta U(\cos(2\theta) - 2)\right)}}{\hbar(2 - \cos(2\theta))}}$$

SOLUTION 2: Center of mass approach

Part I: Conservation laws

First, find the velocity of the center of mass. We get

$$(3mv_{CM}) = p_\gamma = \hbar\omega/c$$

We will use p_γ for calculations

$$v_{CM} = p_\gamma/(3m)$$

We want to relate the relative velocities in CM. We have by magnitude

$$2mv_1 = mv_2$$

And v_2 is the single oxygen atom (the one we don't want). The energy conservation equation is

$$\begin{aligned} E_\gamma = p_\gamma c &= \frac{1}{2}(3m)v_{CM}^2 + \Delta U + \frac{1}{2}mv_2^2 + \frac{1}{2}(2m)v_1^2 \\ &= \Delta U + \frac{p_\gamma^2}{6m} + \frac{1}{2}\times 6m(v_1)^2 \end{aligned}$$

We can now find v_1 in terms of p_γ.

Part 2: geometry and critical condition

We see that the set of possible velocities $\vec{v}_1$ form a circle (they have constant v_1). We need to consider that θ is the angle between $\vec{v}_{CM}$ and $\vec{v}_{CM} + \vec{v}_1$ (this is the full velocity of particle 1).

If the circle of radius v_1 is too small there is no solution (the straight line at the angle will not intersect the circle). If the radius v_1 is big enough, there will always be two solutions intersecting the circle. The critical case (crossover from no solution to two) is when the circle is tangent to the straight line. This is the case we need.

In that case the triangle formed by $0, \vec{v}_{CM}, \vec{v}_{CM} + \vec{v}_1$ is a right triangle.

Warning: This can only happen if $\theta \leq 90^o$. The other case will be dealt with separately.

This way

$$v_1 = v_{CM}\sin(\theta)$$

Replace above to find that in conservation of energy

$$p_\gamma c = \Delta U + \frac{p_\gamma^2}{6m} + \frac{1}{2}\times\frac{p_\gamma^2\sin^2(\theta)}{3m}$$

now we can solve the quadratic equation for p_γ. We get (*we need to choose the correct solution of the quadratic equation*)

$$p_\gamma = \frac{3cm - \sqrt{3}\sqrt{m}\sqrt{3c^2m + 2\Delta U\cos(2\theta) - 4\Delta U}}{2\sin^2(\theta) + 1}$$

Note: Any of the following expressions are the same:

$$2\sin^2(\theta) + 1 = 3 - 2\cos^2(\theta) = 2 - \cos(2\theta)$$

Rewrite as the angular frequency

$$\boxed{\omega_{min} = 3mc^2\left[\frac{1-\sqrt{1-\frac{\Delta U}{3mc^2}(2\sin^2(\theta)+1)}}{(2\sin^2(\theta)+1)\hbar}\right]}$$

Special case: $\theta \geq 90^0$, *this is essentially impossible to see in a brute force algebraic solution, but it is an important feature of the problem. Needs a small amount of points in this solution*

For angle greater than 90^0, as soon as the circle reaches the answer for 90^0 it becomes big enough to allow all angles. Formula is the same with $\theta = 90^0$.

T1-C2 [1.0 pts].

Up to second order in ΔU

$$\omega_{\min} = \frac{\Delta U}{\hbar} - \frac{(\cos(2\theta) - 2)\Delta U^2}{6\hbar mc^2}$$

$$\hbar\omega_{\min} - \Delta U = \frac{(2 - \cos(2\theta))}{6mc^2}(\Delta U)^2$$

$$\boxed{\hbar\omega_{\min} - \Delta U = 2.03 \times 10^{-11}\text{eV}}$$